A Reliable Treatment for Solving Nonlinear Two-Point Boundary Value Problems

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In this paper, we study the modified decomposition method (MDM) for solving nonlinear two-point boundary value problems (BVPs) and show numerical experiments. The modified form of the Adomian decomposition method which is more fast and accurate than the standard decomposition method (SDM) was introduced by Wazwaz. In addition, we will compare the performance of the MDM and the new nonlinear shooting method applied to the solutions of nonlinear two-point BVPs. The comparison shows that the MDM is reliable, efficient and easy for solving the nonlinear two-point BVPs.

Key words: Adomian Polynomials; Nonlinear Two-Point Boundary Value Problems; Padé Approximation.

1. Introduction

In this paper, we consider examples of the nonlinear two-point boundary value problems (BVPs)

$$y'' = f(x,y,y'),\ a \leq x \leq b, \tag{1}$$

with the boundary conditions

$$y(a) = \alpha,\ y(b) = \beta. \tag{2}$$

In general, by two-point BVPs, we mean problems with the following characteristics [1]:

1. n first-order ordinary differential equations have to be solved over the interval $[a,b]$, where a is the initial point and b is the final point;
2. r boundary conditions are specified at the initial value a of the independent variable;
3. $(n-r)$ boundary conditions are specified at the terminal value b of the independent variable.

The following theorem gives general conditions that ensure that the solution to a second-order BVP exists and that it is unique [2].

Theorem. Suppose that the function f in the BVP (1) is continuous on the set

$$D = \{(x,y,y')|a \leq x \leq b, -\infty < y < \infty, -\infty < y' < \infty\},$$

and that $\partial f/\partial x$ and $\partial f/\partial y'$ are also continuous on D. If

1. $\dfrac{\partial f}{\partial y}(x,y,y') > 0$ for all $(x,y,y') \in D$,

and

2. $\left|\dfrac{\partial f}{\partial y'}(x,y,y')\right| < M$ for all $(x,y,y') \in D$,

then the BVP has a unique solution.

Proof. See [2].

The nonlinear two-point BVPs occur in applied mathematics, theoretical physics, engineering, control and optimization theory. If the two-point BVPs can't be solved analytically, which is the usual case, then recourse must be made to numerical methods which are the shooting method, Ritz's method, finite-difference method and Green functions. In recent years, the standard decomposition (SDM) and modified decomposition methods (MDM) have been used by Wazwaz for solving this type of problems.

In this paper, the modified decomposition method is used to investigate the numerical and analytic solutions of the nonlinear two-point BVPs. The modified decomposition method was introduced by Wazwaz [3, 4].

2. The Reliable Method

In the following, we introduce the main features of the standard decomposition and the modified decomposition methods [5].

The principal algorithm of the Adomian decomposition method when applied to a general nonlinear equation is given in the form

$$Lu + Ru + Nu = g. \tag{3}$$

The linear terms are decomposed into $L+R$, while the nonlinear terms are represented by Nu. L is taken as the highest-order derivative to avoid difficult integration involving the complicated Green's functions, and R is the remainder of the linear operator. L^{-1} is regarded as the inverse operator of L and is defined by a definite integration from 0 to t, i. e.,

$$L^{-1}(.) = \int_0^t \int_0^t (.)\,\mathrm{d}t\,\mathrm{d}t. \tag{4}$$

If L is a second-order operator, L^{-1} is a two-fold indefinite integral:

$$L^{-1}Lu = u(x,t) - u(x,0) - t\frac{\partial u(x,1)}{\partial t}|_{t=0}. \tag{5}$$

Operating on both sides of (3) with L^{-1} yields

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{6}$$

and gives

$$\begin{aligned} u(x,t) &= u(x,0) + t\,u_t(x,0) \\ &\quad + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \end{aligned} \tag{7}$$

The decomposition method represents the solution of (7) as a series as follows:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{8}$$

The nonlinear operator Nu is decomposed:

$$Nu = \sum_{n=0}^{\infty} A_n. \tag{9}$$

Substituting (8) and (9) into (7), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{10}$$

where

$$u_0 = u(x,0) + t\,u_t(x,0) + L^{-1}g. \tag{11}$$

Consequently, it can be written as

$$\begin{aligned} u_1 &= -L^{-1}Ru_0 - L^{-1}A_0, \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, n \geq 0, \end{aligned} \tag{12}$$

where A_n are Adomian polynomials of $u_0, u_1, \ldots, u_n$ obtained from the formula

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}F\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots \tag{13}$$

Equation (13) gives

$$\begin{aligned} A_0 &= N(u_0), \\ A_1 &= u_1\frac{\mathrm{d}}{\mathrm{d}u_0}N(u_0), \\ A_2 &= u_2\frac{\mathrm{d}}{\mathrm{d}u_0}N(u_0) + \frac{u_1^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}u_0^2}N(u_0), \\ A_3 &= u_3\frac{\mathrm{d}}{\mathrm{d}u_0}N(u_0) + u_1u_2\frac{\mathrm{d}^2}{\mathrm{d}u_0^2}N(u_0) + \frac{u_1^3}{3!}\frac{\mathrm{d}^3}{\mathrm{d}u_0^3}N(u_0), \\ &\vdots \end{aligned} \tag{14}$$

Recently, Wazwaz [3,4] defined the zeroth component in a slightly different way. He assumed that $u_0 = g$ and the function g can be divided into two parts, such as g_1 and g_2, where a modified recursive scheme can be given as follows:

$$\begin{aligned} u_0 &= g_1, \\ u_1 &= g_2 - L^{-1}Ru_0 - L^{-1}A_0, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n,\ n \geq 0. \end{aligned} \tag{15}$$

This type of modification gives more flexibility to the standard decomposition method in order to solve complicated nonlinear differential equations. In many cases the modified scheme avoids unnecessary computations, especially in the calculation of Adomian polynomials.

The n-term approximant

$$\varphi_n = \sum_{k=0}^{n-1} u_k(x) \tag{16}$$

can be used to approximate the solution. For illustration purposes we will consider the nonlinear two-point BVPs in the following section.

3. Three Examples

In this section, three nonlinear two-point BVPs will be solved by using the modified decomposition method discussed above. In addition, we will present a comparison between our present results and results of Ha [1].

3.1. Example 1

We first consider the BVP

$$y''(x) = y^2(x) + 2\pi^2\cos(2\pi x) - \sin^4(\pi x) \quad \text{for } 0 < x < 1, \tag{17}$$

with the boundary conditions

$$y(0) = y(1) = 0. \tag{18}$$

The problem has the exact solution

$$y_{\mathrm{E}}(x) = \sin^2(\pi x). \tag{19}$$

We can write (17) in an operator form as follows:

$$Ly = y^2(x) + 2\pi^2\cos(2\pi x) - \sin^4(\pi x), \quad 0 < x < 1. \tag{20}$$

Operating with L^{-1} on both sides of (20), and using the boundary condition at $x = 0$ yields

$$y(x) = \alpha x + L^{-1}(y^2(x)) + L^{-1}(2\pi^2\cos(2\pi x) - \sin^4(\pi x)), \tag{21}$$

where

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2},\ L^{-1}(.) = \int_0^x\int_0^{x'}(.)\,\mathrm{d}x\,\mathrm{d}x', \quad \text{and } \alpha = y'(0). \tag{22}$$

Substituting the decomposition series (8) for solution $y(x)$ and the polynomial representation (9) for the nonlinear term $y^2(x)$ into (21), we have

$$\sum_{n=0}^{\infty} y_n(x) = \alpha x + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) + L^{-1}(2\pi^2\cos(2\pi x) - \sin^4(\pi x)). \tag{23}$$

To determine the components $y_n(x)$, $n \geq 0$, we obtain the recursive relation by the MDM:

$$\begin{aligned} y_0(x) &= \alpha x,\\ y_1(x) &= L^{-1}(2\pi^2\cos(2\pi x) - \sin^4(\pi x)) + L^{-1}(A_0),\\ y_{n+1}(x) &= L^{-1}(A_n),\ n \geq 0. \end{aligned} \tag{24}$$

Using the general formula (14), one can generate the Adomian polynomials A_n for the nonlinear term $y^2(x)$ as follows:

$$\begin{aligned} A_0 &= y_0^2,\\ A_1 &= 2y_0y_1,\\ A_2 &= 2y_0y_2 + y_1^2,\\ A_3 &= 2y_1y_2 + 2y_0y_3,\\ A_4 &= 2y_1y_3 + 2y_0y_4 + y_2^2, \cdots . \end{aligned} \tag{25}$$

In view of the recursive relation (24) we obtain

$$\begin{aligned} y_0 &= \alpha x,\\ y_1 &= -\frac{1}{16\pi^2}\big[8\pi^2\cos(2\pi x) - 1 + 5\cos^2(\pi x) - \cos^4(\pi x) + 3\pi^2x^2\big] + 2\alpha^2\frac{x^4}{4!},\\ y_2 &= \frac{1}{645120\pi^5}\big[-161280\pi^2\sin(2\pi x) + 161280\pi^3 x\cos(2\pi x) + 38640\pi^3x^3 - 40320\sin(2\pi x) + 40320\pi x\cos(2\pi x) + \cdots\big]. \end{aligned} \tag{26}$$

Consequently, the approximate solution of $y(x)$ is given by

$$\begin{aligned} y(x) = \alpha x &- \frac{1}{16\pi^2}(8\pi^2\cos(2\pi x) - 1 + 5\cos^2(\pi x) - \cos^4(\pi x) + 3\pi^2x^2)\\ &+ \frac{1}{645120\pi^5}(-161280\pi^2\sin(2\pi x) + 161280\pi^3x\cos(2\pi x) + \cdots). \end{aligned} \tag{27}$$

To determine the constant α, we write the boundary condition at $x = 1$ on the ten-term approximant φ_{10}. Then we get

$$\alpha = 0.000011. \tag{28}$$

The numerical results are given in Table 1.

k,n	$v_0 = 0.25$	$v_0 = 0.50$	$v_0 = 1.0$	$v_0 = 5.0$	$v_0 = 10.0$	Modified technique $\|\varphi_n(x) - y_E(x)\|_\infty$
1	0.2500000	0.5000000	1.000000	5.000000	10.000000	0.0000114
3	0.0000008	0.0000108	0.0002951	0.1326154	1.2200974	0.0000631
5	0.0000008	0.0000008	0.0000000	0.0000000	0.0006724	0.0000035
7	0.0000000	0.0000000	0.0000000	0.0000009	0.0000008	0.0000000

Table 1. Comparison between absolute errors with each initial velocity obtained by the new nonlinear shooting method [3] and by the modified decomposition method for Example 1.

x	$v_0 = 0.25$	$v_0 = 0.50$	$v_0 = 1.0$	$v_0 = 5.0$	$v_0 = 10.0$	Modified technique $\|\varphi_{20}(x) - y_E(x)\|_\infty$
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.000000000
0.05	0.0000002	0.0000002	0.0000002	0.0000002	0.0000002	0.000000061
0.10	0.0000008	0.0000008	0.0000008	0.0000007	0.0000008	0.000000063
0.15	0.0000017	0.0000017	0.0000017	0.0000017	0.0000017	0.000000064
0.20	0.0000028	0.0000028	0.0000028	0.0000028	0.0000028	0.000000064
0.25	0.0000041	0.0000041	0.0000041	0.0000041	0.0000041	0.000000063
0.30	0.0000054	0.0000054	0.0000054	0.0000054	0.0000054	0.000000059
0.35	0.0000066	0.0000066	0.0000066	0.0000066	0.0000066	0.000000049
0.40	0.0000075	0.0000075	0.0000075	0.0000075	0.0000075	0.000000028
0.45	0.0000081	0.0000081	0.0000081	0.0000081	0.0000081	0.000000008
0.50	0.0000083	0.0000083	0.0000083	0.0000083	0.0000083	0.000000006
0.55	0.0000082	0.0000082	0.0000082	0.0000082	0.0000082	0.000000050
0.60	0.0000075	0.0000075	0.0000075	0.0000075	0.0000075	0.000000025
0.65	0.0000066	0.0000066	0.0000066	0.0000066	0.0000066	0.000000038
0.70	0.0000054	0.0000054	0.0000054	0.0000054	0.0000054	0.000000053
0.75	0.0000040	0.0000040	0.0000040	0.0000040	0.0000040	0.000000069
0.80	0.0000027	0.0000027	0.0000027	0.0000027	0.0000027	0.000000087
0.85	0.0000015	0.0000015	0.0000015	0.0000015	0.0000015	0.000000100
0.90	0.0000006	0.0000006	0.0000006	0.0000006	0.0000006	0.000000120
0.95	0.0000002	0.0000002	0.0000002	0.0000002	0.0000002	0.000000140
1.00	0.0000001	0.0000001	0.0000001	0.0000001	0.0000001	0.000000180

Table 2. Approximated errors with each initial velocity for the new nonlinear shooting method [3] and the modified decomposition method (MDM) for Example 1.

All calculations were performed for $x = 0.00, 0.05, \ldots, 1.00$ since the MDM gives good results for small x. Tables 1 and 2 display comparisons of the MDM solution of the nonlinear two-point BVPs with the new shooting method [1] for some values of the initial velocity v_0. As seen in the tables the MDM solutions show the correct physical properties of the problem.

3.2. Example 2

Now we consider the BVP

$$y''(x) = \frac{3}{2}y^2 \text{ for } 0 < x < 1, \tag{29}$$

with the boundary conditions

$$y(0) = 4,\ y(1) = 1. \tag{30}$$

Then the exact solution of (29) is

$$y_E(x) = \frac{4}{(1+x)^2}. \tag{31}$$

Proceeding as before, we obtain the recursive relations

$$\begin{aligned} y_0 &= 4, \\ y_1 &= \beta x + \frac{3}{2}L^{-1}(A_0), \\ y_{n+1} &= \frac{3}{2}L^{-1}(A_n),\ n \geq 0. \end{aligned} \tag{32}$$

In view of (32) we get

$$\begin{aligned} y_0 &= 4, \\ y_1 &= \beta x + 12x^2, \\ y_2 &= 2\beta x^3 + 12x^4, \\ &\vdots \end{aligned} \tag{33}$$

This gives the solution in the form of the series as follows:

$$y(x) = 4 + \beta x + 12x^2 + 2\beta x^3 + 12x^4 + \cdots, \tag{34}$$

where β is a constant. To determine the constant β, we use the boundary condition at $x = 1$ of the approximant φ_3; thus we get

$$\beta = -8.33. \tag{35}$$

To get a better approximation for the constant β, we substitute the boundary condition at $x = 1$ on the Padé approximant [3/5] of the resulting polynomial and obtain

$$\beta = -8.00023. \tag{36}$$

It is clear that we can obtain a sequence of approximations for β by constructing other Padé approximates of other orders [6]. Thus we have

$$\beta = -8. \tag{37}$$

Substituting (37) into (34), we get the solution in the series form as follows:

$$y(x) = 4\left(1 - 2x + 24\frac{x^2}{2!} - 96\frac{x^3}{3!} + \cdots\right), \tag{38}$$

and in the closed form (31).

3.3. Example 3

Finally we consider the nonlinear BVP

$$y''(x) = y^3 - yy' \text{ for } 1 < x < 2, \tag{39}$$

with the boundary conditions

$$y(0) = 1,\ y(1) = \frac{1}{2}. \tag{40}$$

The exact solution of (39) is

$$y_{\mathrm{E}}(x) = \frac{1}{1+x}. \tag{41}$$

Equation (41) can be given in the operator form as

$$Ly = y^3 - yy' \text{ for } 1 < x < 2. \tag{42}$$

Operating with L^{-1} on (42), and using the boundary condition at $x = 0$, we obtain

$$y(x) = 1 + \gamma x + L^{-1}(y^3(x) - y(x)y'(x)), \tag{43}$$

where the inverse operator L^{-1} is a two-fold integral operator and the constant

$$\gamma = y'(0) \tag{44}$$

must be determined. Substituting the decomposition series (8) for $y(x)$ and the series of polynomials (9) gives

$$\sum_{n=0}^{\infty} y_n(x) = 1 + \gamma x + L^{-1}\left(\sum_{n=0}^{\infty} B_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} C_n\right). \tag{45}$$

The first few Adomian polynomials B_n and C_n are given by

$$\begin{aligned} B_0 &= y_0^3,\\ B_1 &= 3y_0^2 y_1,\\ B_2 &= 3y_0^2 y_2 + 3y_0 y_1^2,\\ B_3 &= y_1^3 + 3y_0^2 y_3 + 6y_0 y_1 y_2,\\ &\vdots \end{aligned} \tag{46}$$

$$\begin{aligned} C_0 &= y_0 y_0',\\ C_1 &= y_1 y_0' + y_0 y_1',\\ C_2 &= y_2 y_0' + y_1 y_1' + y_0 y_2',\\ C_3 &= y_3 y_0' + y_2 y_1' + y_1 y_2' + y_0 y_3',\\ &\vdots \end{aligned} \tag{47}$$

Proceeding as before we get the recurrence relation

$$\begin{aligned} y_0 &= 1,\\ y_1 &= \gamma x + L^{-1}(B_0) - L^{-1}(C_0),\\ y_{n+1} &= L^{-1}(B_n) - L^{-1}(C_n),\ n \geq 0. \end{aligned} \tag{48}$$

Substituting (46) and (47) into (48), we have

$$\begin{aligned} y_0 &= 1,\\ y_1 &= \gamma x + \frac{x^2}{2!},\\ y_2 &= -\gamma\frac{x^2}{2!} + (3\gamma - 1)\frac{x^3}{3!} + 3\frac{x^4}{4!},\\ y_3 &= (\gamma - \gamma^2)\frac{x^3}{3!} + (6\gamma^2 - 9\gamma + 1)\frac{x^4}{4!}\\ &\quad + (27\gamma - 9)\frac{x^5}{5!} + 24\frac{x^6}{6!}. \end{aligned} \tag{49}$$

Other components can be determined in the same way. This gives the solution in a series form:

$$\begin{aligned} y(x) &= 1 + \gamma x + (1-\gamma)\frac{x^2}{2!} + (-\gamma^2 + 4\gamma - 1)\frac{x^3}{3!}\\ &+ (6\gamma^2 - 9\gamma + 4)\frac{x^4}{4!} + (27\gamma - 9)\frac{x^5}{5!} + 24\frac{x^6}{6!} + \cdots \end{aligned} \tag{50}$$

To determine the constant γ, we write the boundary condition at $x = 1$ on the four-term approximant φ_4; then we get

$$\gamma = -1.108. \tag{51}$$

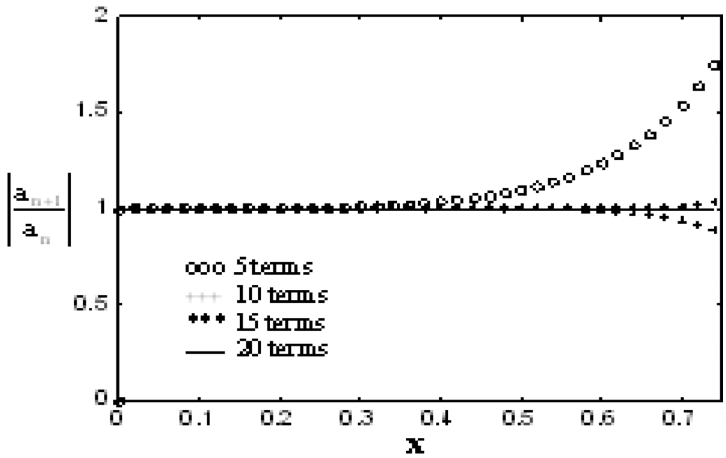

Fig. 1. The ratio convergence test applied to the series coefficients (54).

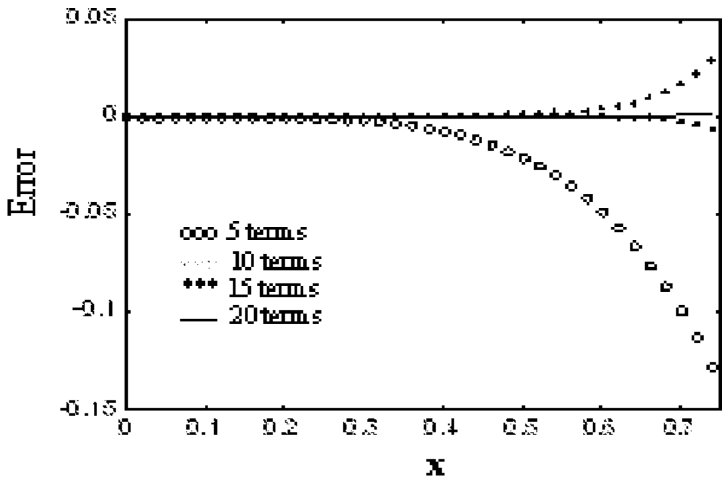

Fig. 2. Differences between the MDM solution and the exact solution of (39).

To determine a better approximation for the constant γ, we substitute the boundary condition at $x = 1$ on the Padé approximant [3/5] of the resulting polynomial and obtain

$$\gamma = -1.0001. \tag{52}$$

It is clear that we can obtain a sequence of approximations for γ by constructing other Padé approximates of other orders [6]. Thus we have

$$\gamma = -1. \tag{53}$$

Substituting (53) into (51), we get the solution in a series form:

$$y(x) = 1 - x + x^2 - x^3 + x^4 - \cdots, \tag{54}$$

and in the closed form (41).

4. Convergence of the Solution in Series Form

The decomposition method provides an analytic solution in terms of an infinite power series. The analytical solution given in (38) and (54) can be expressed in the series form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{55}$$

The series (55) consists of both positive and negative terms, although not in a regular alternating fashion. The ratio test was applied to the absolute values of the series coefficient. This provides a sufficient condition for convergence of the series for a space interval $\Delta x = x_b - x_a$, which has the form

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < \frac{1}{\Delta x}. \tag{56}$$

It is clear from Fig. 1 that the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ decayes as n increases, obviously indicating that the series (55) is convergent.

We can also draw (38) in a like manner. In order to investigate the accuracy of the modified decomposition solution with a finite number of terms (see Fig. 2), (17), (29) and (39) were also solved numerically by Ha [1], and the corresponding results are compared with the decomposition solution in Tables 1 and 2. The numerical method adopted in [1] was the new nonlinear shooting method, which was used in the fourth-order Runge-Kutta method and Newton's method with an error bound of 10^{-7}. Ha obtained numerical solutions with some initial velocities converging to an exact solution, but his numerical solutions with some special choice of initial velocities did not converge to an exact solution. However, we obtained the exact solution for (29) and (39) by our present method. The convergence of the modified decomposition method is very rapidly. Therefore, it may be concluded that the use of 15 terms in the series yields sufficiently accurate solutions for values close to zero.

5. Concluding Remarks

In this paper, we calculated the approximate solutions of some nonlinear two-point boundary value problems by using the modified decomposition method (MDM). We demonstrated that the decomposition procedure is quite efficient in determining solutions in the closed form by using boundary conditions. Our present method avoids the tedious work needed by traditional techniques. We got more accurate approximate solutions by using the boundary conditions of the MDM in Example 1 and we obtained the analytic solutions by this method in Examples 2 and 3.

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